THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
HW9 Solution

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1. (P.252 Q7) Since (f_n) converges uniformly to f on A, choose $\epsilon = 1$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\|f_n - f\|_A < 1$. In particular, consider $n = N$, then by assumption there exists $M_N \in \mathbb{R}$ such that for all $x \in A$, $|f_N(x)| \leq M_N$. Therefore, for all $x \in A$, $|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < 1 + M_N$. Therefore, f is bounded on A.

2. (P.252 Q8) For each $n \in \mathbb{N}$, we claim that $f_n(x)$ is bounded on $[0, +\infty)$: on $[0, 1]$,

$$|f_n(x)| = |\frac{nx}{1 + nx^2}| \leq n$$

on $[1, +\infty)$,

$$|f_n(x)| = |\frac{nx}{1 + nx^2}| \leq |\frac{nx^2}{1 + nx^2}| < 1$$

Therefore, for all $x \in [0, +\infty)$, $|f_n(x)| \leq n$, and hence f_n is bounded for each $n \in \mathbb{N}$.

Fix each $x \in [0, +\infty)$, then $\lim_{n\to\infty} \frac{nx}{1 + nx^2} = \lim_{n\to\infty} \frac{x}{\frac{1}{n} + x^2} = \begin{cases} 0 & x = 0 \\ \frac{1}{x} & x \neq 0 \end{cases}$

Therefore, the pointwise limit of (f_n) is given by $f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x} & x \neq 0 \end{cases}$. Since $\lim_{x\to 0^+} f(x) = +\infty$, f is not bounded on $[0, \infty)$.

If (f_n) converges uniformly to f on $[0, +\infty)$, then by the result of Q7, f is also bounded on $[0, +\infty)$, which is a contradiction. Therefore, (f_n) does not converge uniformly to f on $[0, +\infty)$.

3. (P.252 Q12) We first show that $f_n(x) = e^{-nx^2}$ converges uniformly to 0 on $[1, 2]$: since $e^{nx^2} \geq nx^2 \geq n$ for all $n \in \mathbb{N}$ and $x \in [1, 2]$, $|f_n(x) - 0| = e^{-nx^2} \leq \frac{1}{n}$. Therefore, $\|f_n\|_{[1,2]} \leq \frac{1}{n} \to 0$ as $n \to \infty$. Therefore, by Lemma 8.1.8, $f_n(x) = e^{-nx^2}$ converges uniformly to 0 on $[1, 2]$.

Therefore, by Theorem 8.2.4, $\lim_{n\to\infty} \int_1^2 e^{-nx^2} dx = \int_1^2 0 dx = 0$.